Optimal shape of stellarators for magnetic confinement fusion.

In collaboration with Yannick Privat [1] and Mario Sigalotti[2]

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Nuclear fusion confinement

- Goal: Confine a plasma of approx. 150 millions K for as long as possible with a density as high as possible in order to achieve fusion ignition.

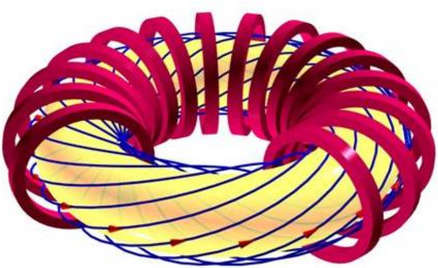

Figure: magnetic field lines inside a Tokamac, Inria team TONUS

Nuclear fusion confinement

- Goal: Confine a plasma of approx. 150 millions K for as long as possible with a density as high as possible in order to achieve fusion ignition.
- Solution: a plasma is made of ionized particules, thus interacts with a magnetic field.

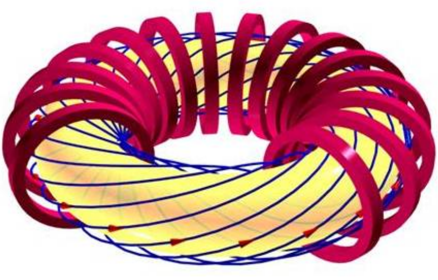

Figure: magnetic field lines inside a Tokamac, Inria team TONUS

Stellarators

Stellarator approach: The magnetic confinement relies mainly on external coils.

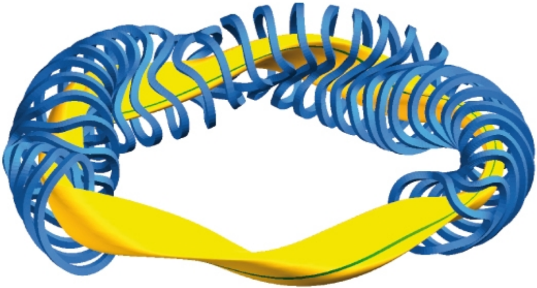

Figure: Wendelstein 7-X, Max-Planck Institut für Plasmaphysik

The plasma shape and the coils are obtained by several optimizations.

Typical approach

1. Find a good magnetic field to ensure the plasma confinement.

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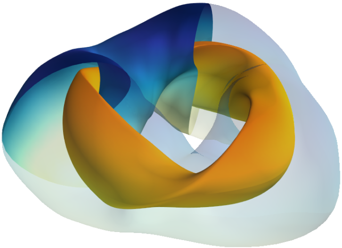
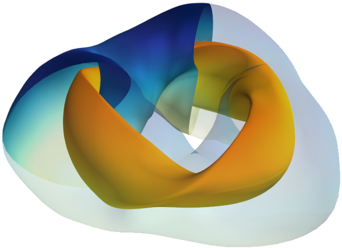


Figure: Coil winding surface and plasma surface of the NCSX Stellarator.

Typical approach

1. Find a good magnetic field to ensure the plasma confinement.
2. We use a 'Coil winding surface' and find a current-sheet to generate the given B_{target} [Merkel 86].
3. (Approximate the current-sheet by several coils)



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Modelisation

Biot-Savart law in vacuo

$$\forall y \notin S, B(y) = \mathrm{BS}(j)(y) = \int_S j(x) \times \frac{y - x}{|y - x|^3} dS(x), \quad (1)$$

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plasma-shape objective

$$\chi_B^2(j) = \int_P |\mathrm{BS}(j)(y) - B_{target}(y)|^2 dy \quad (2)$$

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The goal

$$\inf_{\substack{j \in L^2(\mathfrak{X}(S)) \\ \operatorname{div} j = 0}} \chi_B^2(j) \tag{3}$$

An inverse problem

$BS(\cdot)$ is continuous from $L^2(\mathfrak{X}(S)) \to C^k(P, \mathbb{R}^3)$
$\implies j \mapsto BS(j)$ is compact (from $L^2(\mathfrak{X}(S)) \to L^2(P, \mathbb{R}^3)$).

- Use a finite dimensional subspace [Merkel 86].
- Use a Tychonoff regularization [Landreman 17].

$$\|j\|_{L^2}^2 = \int_S |j|^2 dS.$$

Lemma

For any $\lambda > 0$, the problem

$$\inf_{\substack{j \in L^2(\mathfrak{X}(S)) \\ \operatorname{div} j = 0}} \chi_B^2 + \lambda \|j\|_{L^2}^2 \tag{P}$$

admits a unique minimizer

$$j_S = (\lambda \, \mathrm{Id} + \mathrm{BS}_S^\dagger \, \mathrm{BS}_S)^{-1} \, \mathrm{BS}_S^\dagger \, B_T. \tag{4}$$

Table of Contents

We want to optimize on both the current sheet and the Coil Winding Surface.

Admissible shapes

- Topology of a torus
- Regular enough
- Far enough to the plasma

Shape optimization problem

$$\inf_{S \text{ admissible}} \left(\inf_{\substack{j \in L^2(\mathfrak{X}(S)) \\ \operatorname{div} j = 0}} \chi_B^2 + \lambda \|j\|_{L^2}^2 \right) \quad \text{(SOP)}$$

Previous works

First approach in E. J. Paul et al. "An adjoint method for gradient-based optimization of stellarator coil shapes". In: *Nuclear Fusion* 58.7 (2018)

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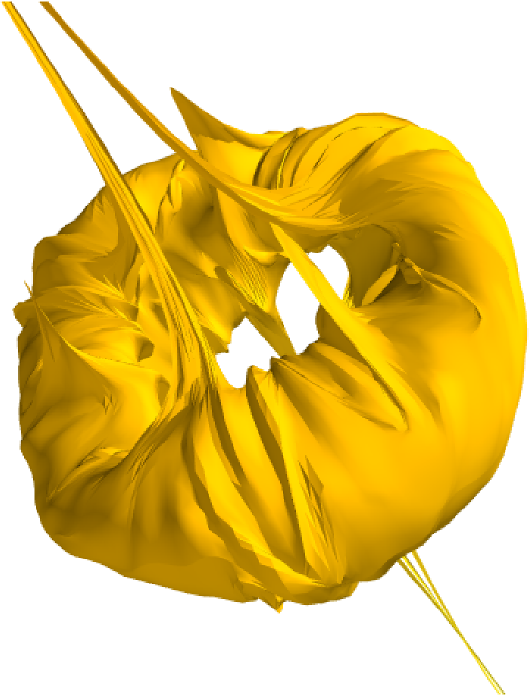
Our contribution

- Existence of a minimizer of the shape optimisation problem,
- Computation of the shape gradient in the set of admissible shapes,
- Numerics based on our approach.

Admissible shapes

Constraints on the set of admissible shapes $S \in \mathcal{O}_{\text{adm}}$:

1. S is a orientable surface homotopic to the usual torus.
2. $dist(S, P) \geq \delta$
3. S is in included in a compact set
4. $\mathcal{H}^2(S) \leq A_M$



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5. Lower bound on the reach of S

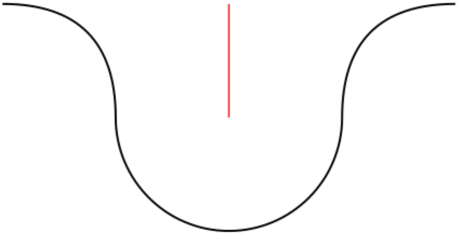
Reach

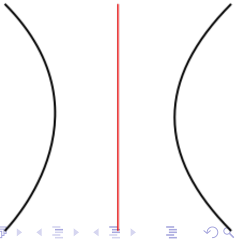
Delfour-Zolesio and Federer

$V \subset \mathbb{R}^n$, $\mathrm{Sk}(V)$ is the set of all points in $\mathbb{R}^n$ whose projection onto V is not unique.

$$U_h(V) = \{x \mid d(x, V) < h\}$$

$$\mathrm{Reach}(V) = \sup\{h \mid U_h(V) \cap \mathrm{Sk}(V) = \emptyset\}$$





Reach

Theorem[2022, Privat, R. , Sigalotti]

The shape optimisation problem

$$\inf_{S \in \mathcal{O}_{\text{adm}}} \inf_{\substack{j \in L^2(\mathfrak{X}(S)) \\ \operatorname{div} j = 0}} \chi_B^2 + \lambda \|j\|_{L^2}^2 \tag{5}$$

admits a minimizer.

Key ingredients of the proof:

- Compactness of $\mathcal{O}_{\text{adm}}$,
- Lower semicontinuity of the cost.
  - Transport j while preserving tangent and divergence free,
  - Use a volumic approximation.

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Theorem[2022, Privat, R. , Sigalotti]

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$\Longrightarrow$ Development of a general framework in [PRS22a]

Shape gradient tool

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- Nevertheless the range of $\mathscr{F}_S^0$ by φ^ε does not coincide with $\mathscr{F}_{S^\varepsilon}^0$.

$$\begin{aligned}\Phi^\varepsilon : \mathscr{F}_S &\longrightarrow \mathscr{F}_{S^\varepsilon} \\ X &\longmapsto \frac{1}{[J(\mu_S, \mu_S^\varepsilon)\varphi^\varepsilon] \circ \varphi^{-\varepsilon}}(\operatorname{Id} + \varepsilon D\theta)X \circ \varphi^{-\varepsilon}\end{aligned}$$

Shape gradient

$$
\begin{aligned}
Z_P(k) &= \int_P K(\cdot, y) \times k(y)\, d\mu_P(y) \\
\widehat{Z}_P(k, j)(x) &= \int_P D_x \left(\frac{x-y}{|x-y|^3} \right)^T (k(y) \times j(x))\, d\mu_P(y), \qquad \forall x \in S.
\end{aligned}
$$

For every $\theta \in W^{2,\infty}(\mathbb{R}^3, \mathbb{R}^3)$ one has

$$
\boxed{\langle dC(S), \theta \rangle = \int_S \theta \cdot (X_1 - \mathrm{div}_S (X_2)_{i:})\, d\mu_S}
$$

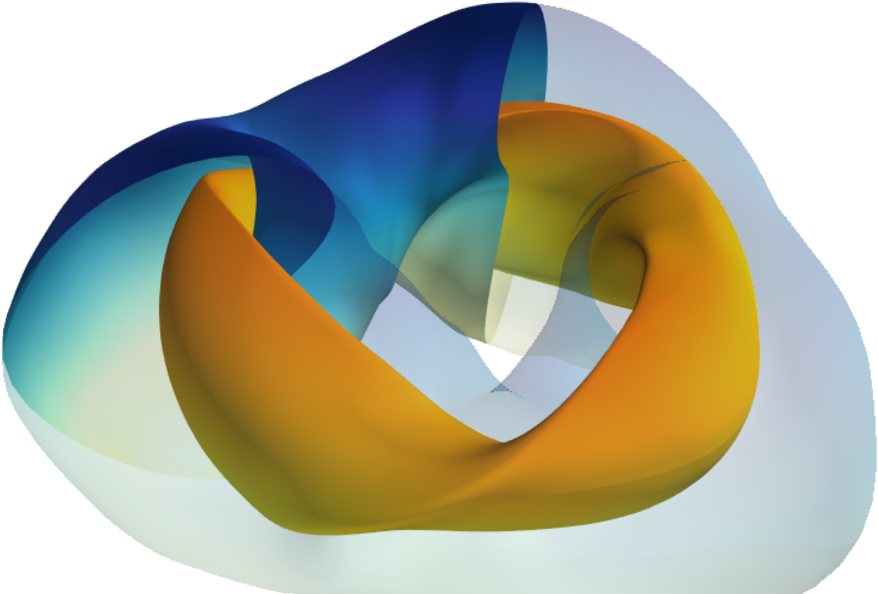
where

$$
X_1 = -2\widehat{Z}_P(\mathrm{BS}_S\, j_S - B_T, j_S) \tag{6}
$$

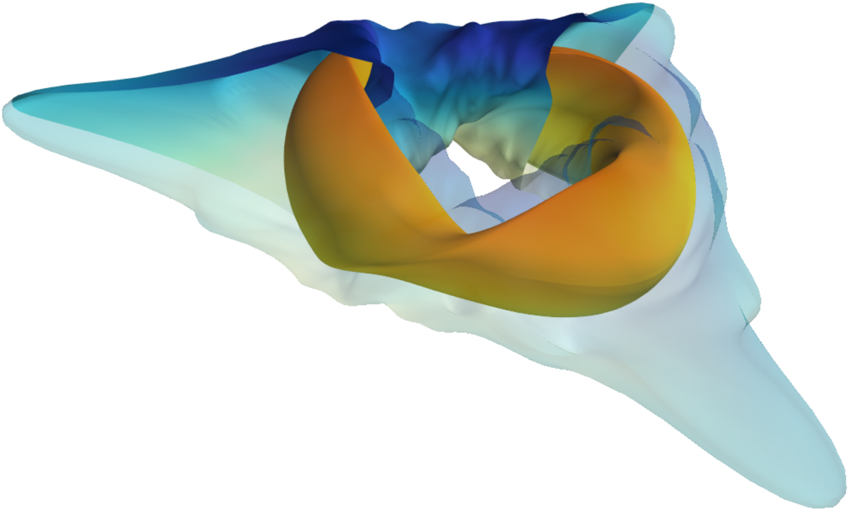
$$
X_2 = -2Z_P(\mathrm{BS}_S\, j_S - B_T) j_S^T + 2\lambda j_S j_S^T - \lambda |j_S|^2 (I_3 - \nu\nu^T), \tag{7}
$$

where for $i \in \{1, 2, 3\}$, $(X_2)_{i:}$ denotes the i-th line of X_2 seen as a column vector, and ν denotes the outward normal vector on $S = \partial V$.

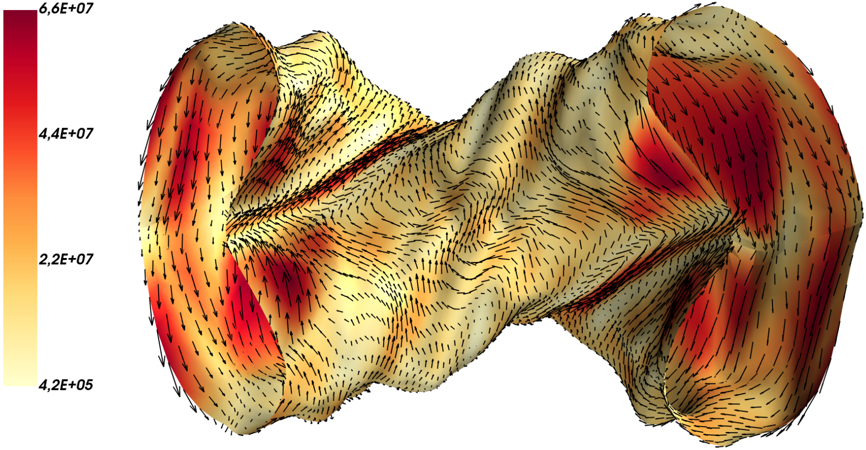
Reference



After optimization



For a different λ



Prospects

- Collaboration with Renaissance fusion[3] for industrial applications,
- Use more complex costs in the shape optimization (Laplace forces).

Thank you for your attention !

[3] https://stellarator.energy/

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Cohomology and divergence free vector fields on the torus

Hodge decomposition

On a closed Riemannian manifold M

$$L^2_p(M) = B_p \oplus B_p^* \oplus \mathscr{H}_p,$$

where

- B_p is the L^2-closure of $\{d\alpha \mid \alpha \in \Omega^{p-1}(M)\}$,
- B_p^* is the L^2-closure of $\{d^*\beta \mid \beta \in \Omega^{p+1}(M)\}$ (d^* is the coderivative),
- $\mathscr{H}_p$ is the set $\{\omega \in \Omega^p(M) \mid \Delta_H \omega = 0\}$ of harmonic p-forms with Δ_H the Hodge Laplacian.

Thus for a flat Torus T, we only need to characterizes $B_1^*(T)$ and $\mathscr{H}_1(T)$.

- $B_1^*(T)$ is the L^2-closure of the 1-forms $\frac{\partial \Phi}{\partial u} dv - \frac{\partial \Phi}{\partial v} du$ for $\Phi \in \mathscr{C}^\infty(T)$.
- $\mathscr{H}_1(T)$ is a two-dimensional vector space as $b_1 = 2$.
  $\mathscr{H}_1(T) = \{\lambda_1 du + \lambda_2 dv \mid (\lambda_1, \lambda_2) \in \mathbb{R}^2\}$.

In vacuo Maxwell equations on a toroidal 3D domain

Let P a be toroidal domain. Let Γ be a toroidal loop inside P and denote by I_p the electric current-flux across any surface enclosed by Γ (also equal to the circulation of B along Γ).

Lemma

Let $B \in \mathcal{C}^\infty(P, \mathbb{R}^3)$ such that $\operatorname{div} B = 0$ and $\operatorname{curl} B = 0$ in P. Let g be the normal magnetic field on ∂P. Then g and I_p determine completely the magnetic field B in P. Besides, there exists a constant $C > 0$ such that for every other magnetic field $\tilde{B}$ with the same total poloidal currents, $|B - \tilde{B}|_{L^2(P,\mathbb{R}^3)} \leq C|g - \tilde{g}|_{L^2(\partial P)}$ where $\tilde{g}$ is the normal component of $\tilde{B}|_{\partial P}$.

Idea: consider the cochain complex

$$\mathscr{C}^\infty(P) \xrightarrow{\text{grad}} \mathscr{C}^\infty(P, \mathbb{R}^3) \xrightarrow{\text{curl}} \mathscr{C}^\infty(P, \mathbb{R}^3) \xrightarrow{\text{div}} \mathscr{C}^\infty(P).$$