On the Laplace forces on a current-sheet.

This is a work in collaboration with Francesco Volpe[1] and Mario Sigalotti[2]

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Nuclear fusion confinement

- Goal : Confine a plasma of approx. 150 millions K for as long as possible with a density as high as possible in order to achieve fusion ignition.
- Solution : A plasma is a made of ionized particules, thus interacts with a magnetic field.

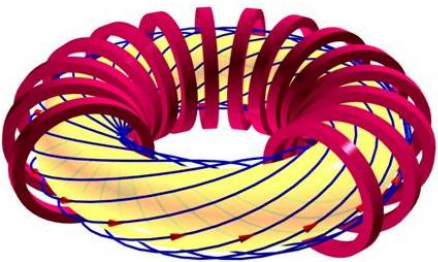

Figure: magnetic field lines inside a Tokamac, Inria team TONUS

Stellarators

Stellarator approach : The magnetic confinement relies mainly on external coils.

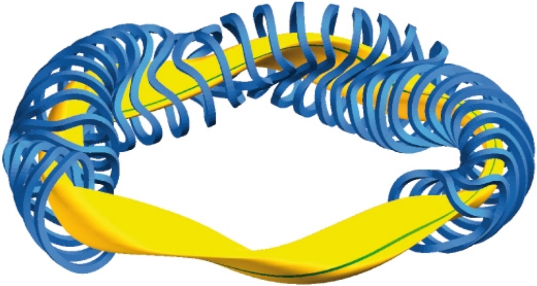

Figure: Wendelstein 7-X, Max-Planck Institut für Plasmaphysik

The plasma shape and the coils are obtained by several optimizations.

Typical approach

1. Find a good magnetic field to ensure the plasma confinement. On the Plasma boundary, B_{target} is tangent to the surface. This surface characterizes (nearly) entirely the magnetic field.

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2. We use a 'Coil winding surface' and find a current-sheet to generate the given B_{target}.

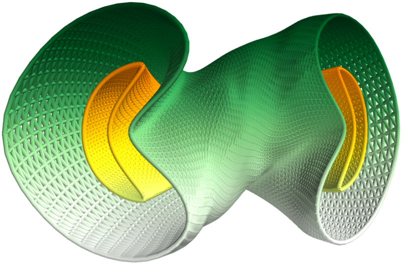

Figure: Coil winding surface and plasma surface of the NCSX Stellarator.

Typical approach

1. Find a good magnetic field to ensure the plasma confinement. On the Plasma boundary, B_{target} is tangent to the surface. This surface characterizes (nearly) entirely the magnetic field.
2. We use a 'Coil winding surface' and find a current-sheet to generate the given B_{target}.
3. (Approximate the current-sheet by several coils)

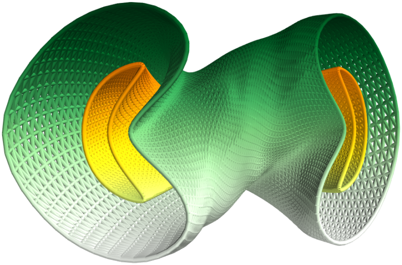

Figure: Coil winding surface and plasma surface of the NCSX Stellarator.

B is (in good approximation) only generated by electric currents on the CWS (denoted S).

Biot-Savart law in vacuo

$$\forall y \notin S, B(y) = \mathrm{BS}(j)(y) = \int_S j(x) \times \frac{y-x}{|y-x|^3} dS(x), \tag{1}$$

The figure of merit we use to ensure $B \approx B_{target}$ is

plasma-shape objective

$$\chi_B^2 = \int_{S_P} \langle B(x) \cdot n(x) \rangle^2 dS(x). \tag{2}$$

For a nice closed affine subspace E of $L^2(\mathfrak{X}(S))$

Inverse problem

$$\inf_{j \in E} \chi_B^2 \tag{P}$$

An inverse problem

$BS(\cdot)$ is continuous from $L^2(\mathfrak{X}(S)) \to C^k(S_P, \mathbb{R}^3)$
$\implies j \mapsto \langle BS(j) \cdot n \rangle$ is compact (from $L^2(\mathfrak{X}(S)) \to L^2(S_P)$).

- Use a finite dimensional subspace for the space of vector field to solve the problem and use the dimension as a regularization parameter. See `NESCOIL` [3].
- Use a Tychonoff regularization $\lambda \chi_j^2$ to ensure existence of the minimizer. This is done in `REGCOIL` code [2].

$$\chi_j^2 = \int_S |j|^2 dS. \tag{3}$$

Lemma

For any $\lambda > 0$, the problem

$$\inf_{j \in E} \chi_B^2 + \lambda \chi_j^2 \tag{P}$$

admit a unique minimizer.

About Poisson equation and some cohomology

Let P a full 3D torus

$$\begin{array}{ccccccc}
\Omega^0(P) & \xrightarrow{d} & \Omega^1(P) & \xrightarrow{d} & \Omega^2(P) & \xrightarrow{d} & \Omega^3(P) \\
\Big\downarrow Id & & \Big\downarrow \# & & \Big\downarrow \beta^{-1} & & \Big\downarrow * \\
C^\infty(P) & \xrightarrow{\text{grad}} & \mathfrak{X}(P) & \xrightarrow{\text{curl}} & \mathfrak{X}(P) & \xrightarrow{\text{div}} & C^\infty(P)
\end{array}$$

$b_0 = 1, b_1 = 1, b_2 = 0$. As $b_1 = 1$, Dim $Ker\,\text{curl}\,/\,Im\,\text{grad} = 1$. Besides by Hodge decomposition there exists $X \in \mathfrak{X}(S)$ such that

- $X \notin Im(\text{grad})$
- $\text{curl}\,X = 0$
- $\text{div}\,X = 0$

e.g. $X = \frac{e_\theta}{r}$

$$\text{curl}\, B = 0 \text{ on P} \implies \exists f \in C^\infty(P), \exists \nu > 0, \text{ s.t. } B = \text{grad}\, f + \nu X. \quad (4)$$

Let Γ be a loop of index 1 and S_Γ any surface enclosed by Γ. The line integral of B along Γ is given by the total poloidal current $I_p = \iint_{S_\Gamma} j \cdot \vec{da}$.

$$I_p = \oint_\Gamma B \cdot \vec{dl} = \oint_\Gamma (\text{grad}\, f + \nu X) \cdot \vec{dl} = \nu \oint_\Gamma X \cdot \vec{dl} \quad (5)$$

Thus for a given S_p, $I_p \to \nu$.

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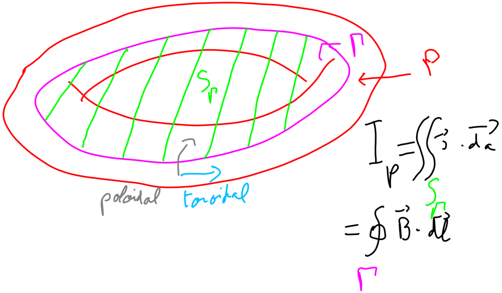


Thus for a given S_p, $I_p \to \nu$.

$$\text{div}\, B = 0 \implies \Delta f = 0 \text{ in } P \quad (5)$$

$$B \cdot n = 0 \text{ on } S_P = \partial P \implies \partial_n f + \nu \langle X \cdot n \rangle = 0 \text{ in } \partial P \quad (6)$$

About divergence-free vector field on a 2D manifold

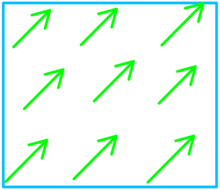
Divergence-free vector field on a flat Torus

Let $T = (\mathbb{R}/\mathbb{Z})^2$ the flat torus with cartesian parametrization (θ, φ). Let $X \in \mathfrak{X}(T)$, then the following proposition are equivalent:

- $\operatorname{div} X = 0$
- $\exists \Phi \in C^\infty(T), \exists (p, q) \in \mathbb{R}^2,$ s.t. $X = \nabla^\perp \Phi + p\partial_\theta + q\partial_\varphi$

with $\nabla^\perp \Phi = \frac{\partial f}{\partial \theta}\partial_\varphi - \frac{\partial f}{\partial \varphi}\partial_\theta$

In practice, we fix p and q and look for Φ which we developped on Fourier series.



Preservation of divergence-free vector field

Let $\psi : T \to S \subset \mathbb{R}^3$ a diffeomorphism, and

$$\tilde{\psi} : \mathfrak{X}(T) \to \mathfrak{X}(S) \tag{7}$$

$$X \mapsto \frac{d\psi X}{|d\psi\partial_\theta \wedge d\psi\partial_\varphi|} \tag{8}$$

Then $\tilde{\psi}$ is a diffeomorphism between $\{X \in \mathfrak{X}(T) \mid \operatorname{div} X = 0\}$ and $\{X \in \mathfrak{X}(S) \mid \operatorname{div} X = 0\}$

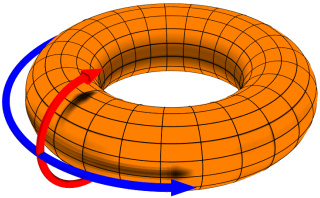


Figure: poloidal (red, θ) and toroidal (blue, φ).

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Motivations

- Building a Stellarator is expensive. . .

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- Higher magnetic fields call for higher currents
- $\implies$ The Laplace forces ($d\vec{F} = i\vec{dl} \wedge \vec{B}$) grew quadratically.

$\implies$ The Laplace forces must be optimized.

Problem

How can we define the Laplace forces on a current-sheet?

Statement of the problem

Let S a toroidal surface and $j \in \mathfrak{X}(S)$ a vector field.

Biot and Savart

$$\forall y \notin S, B(y) = \mathrm{BS}(j)(y) = \int_S j(x) \times \frac{y-x}{|y-x|^3} dS(x),$$

Not integrable

B is not defined on S, indeed for any $y \in S$,

$$\int_S \frac{1}{|x-y|^2} dx = \infty$$

There is a magnetic discontinuity on the surface given by

$$B_T^1 - B_T^2 = n_{12} \wedge j.$$

About the Laplace forces

- B does not blow up near S.
- The discontinuity of B is responsible for a normal force proportional to $|j|^2$ trying increase the thickness of S.

Average Laplace forces

We focus on the other contributions of the Laplace forces, and therefore we define:

$$L_\varepsilon(j)(y) = \frac{1}{2}\left(j \wedge \left[B(j)(y + \varepsilon n(y)) + B(j)(y - \varepsilon n(y))\right]\right)$$

$$L(j) = \lim_{\varepsilon \to 0} L_\varepsilon(j)$$

This definition raises several questions:

1. Under which assumptions on j can we ensure that $L(j)$ is well defined?
2. Can we find an explicit expression of $L(j)$ (i.e. without a limit on ε)?
3. Which functional space does $L(j)$ belong to (for j in a given functional space)?

A 3 scales problem

To compute L from L_ε, we need 3 scales :

1. the discretisation-length of S : h,
2. the infinitesimal displacement ε,
3. the characteristic distance of variation of the magnetic field, d_B.

With :

- $h \ll \varepsilon$ as $\int_S |y + \varepsilon n(y) - x|^{-2} dS(x)$ blows up when $\varepsilon \to 0$.
- $\varepsilon \ll d_B$ to approximate L.

Theorem

Suppose $j_1, j_2 \in \mathfrak{X}^{1,2}(S)$, then $L_\varepsilon(j_1, j_2)$ has a limit in $L^p(S, \mathbb{R}^3)$ for any $1 \leq p < \infty$ when $\varepsilon \to 0$, denoted $L(j_1, j_2)$. Besides, L is a continuous bilinear map $\mathfrak{X}^{1,2}(S) \times \mathfrak{X}^{1,2}(S) \to L^p(S, \mathbb{R}^3)$ given by

$$
\begin{aligned}
L(j_1, j_2)(y) = &- \int_S \frac{1}{|y-x|}\left[\operatorname{div}_x(\pi_x j_1(y)) + \pi_x j_1(y) \cdot \nabla_x\right] j_2(x)dx && (9)\\
&+ \int_S \langle j_1(y) \cdot n(x)\rangle \frac{\langle y-x, n(x)\rangle}{|y-x|^3} j_2(x)dx && (10)\\
&+ \int_S \frac{1}{|y-x|}\left[\langle j_1(y) \cdot j_2(x)\rangle \operatorname{div}_x(\pi_x) + \nabla_x \langle j_1(y) \cdot j_2(x)\rangle\right] dx && (11)\\
&- \int_S \langle j_1(y) \cdot j_2(x)\rangle \frac{\langle y-x, n(x)\rangle}{|y-x|^3} n(x)dx && (12)
\end{aligned}
$$

Some ideas of the proof

- Use $A \wedge (B \wedge C) = (A \cdot C)B - (A \cdot B)C$
- Note that $\frac{y-x}{|y-x|^3} = -\nabla_x \frac{1}{|y-x|}$.
- Do an integration by part on the tangential component of the gradient.
- Use some estimates when ε is small to eliminate the part responsible for the magnetic discontinuity.
- Tools : Hardy-Littlewood-Sobolev inequality and Sobolev embeding on compact manifold [1].

$$L_\varepsilon(j_1,j_2)(y) = \int_S \langle j_1(y)\cdot\big(\frac{y-x+\varepsilon n(y)}{2|y-x+\varepsilon n(y)|^3} + \frac{y-x-\varepsilon n(y)}{2|y-x-\varepsilon n(y)|^3}\big)\rangle j_2(x)dx$$
$$- \int_S \langle j_1(y)\cdot j_2(x)\rangle\big(\frac{y-x+\varepsilon n(y)}{2|y-x+\varepsilon n(y)|^3} + \frac{y-x-\varepsilon n(y)}{2|y-x-\varepsilon n(y)|^3}\big)dx.$$

$$\int_S \langle j_1(y)\cdot \frac{y-x\pm\varepsilon n(y)}{|y-x\pm\varepsilon n(y)|^3}\rangle j_2(x)dx \tag{13}$$
$$= \int_S \langle j_1(y)\cdot \nabla_x \frac{1}{|y-x\pm\varepsilon n(y)|}\rangle j_2(x)dx \tag{14}$$
$$= \int_S \langle j_1(y)\cdot \nabla_S \frac{1}{|y-x\pm\varepsilon n(y)|}\rangle j_2(x)dx \tag{15}$$
$$+ \int_S \langle j_1(y)\cdot \frac{\langle y-x, n(x)\rangle \pm \varepsilon\langle n(y), n(x)\rangle}{|y-x\pm\varepsilon n(y)|^3} n(x)\rangle j_2(x)dx \tag{16}$$

Tangential terms

Integration by part

$$\int_{\mathcal{M}} \operatorname{div}(fX) = 0 = \int_{\mathcal{M}} Xf + f \operatorname{div} X$$

$$\int_S \langle j_1(y) \cdot \nabla_S \frac{1}{|y - x \pm \varepsilon n(y)|} \rangle_{\mathbb{R}^3} j_2(x) dx = \int_S \langle \pi_x j_1(y) \cdot \nabla_S \frac{1}{|y - x \pm \varepsilon n(y)|} \rangle_{T_x S} j_2(x) dx \tag{17}$$

Then, let $j_2^i(x)$ be the i-th component in $\mathbb{R}^3$ of j_2. the i-th component of (17) writes

$$\int_S \langle j_2^i(x) \pi_x j_1(y) \cdot \nabla_S \frac{1}{|y - x \pm \varepsilon n(y)|} \rangle_{T_x S} dx \tag{18}$$

$$= - \int_S \frac{1}{|y - x \pm \varepsilon n(y)|} \operatorname{div}_x (j_2^i(x) \pi_x j_1(y)) dx \tag{19}$$

$$= - \int_S \frac{1}{|y - x \pm \varepsilon n(y)|} \left[j_2^i(x) \operatorname{div}_x(\pi_x j_1(y)) + \langle \pi_x j_1(y) \cdot \nabla j_2^i(x) \rangle \right] dx \tag{20}$$

normal terms

$$\int_S \langle j_1(y) \cdot j_2(x) \rangle \frac{\langle y - x, n(x) \rangle}{|y - x \pm \varepsilon n(y)|^3} dx \pm \int_S \langle j_1(y) \cdot j_2(x) \rangle \frac{\varepsilon \langle n(y), n(x) \rangle}{|y - x \pm \varepsilon n(y)|^3} dx \quad (21)$$

which converges to

$$\int_S \langle j_1(y) \cdot j_2(x) \rangle \frac{\langle y - x, n(x) \rangle}{|y - x|^3} dx,$$

Lemma

$\exists C > 0, \forall x \neq y \in S, \frac{|\langle y - x, n(x) \rangle|}{|y - x|^2} \leq C.$

Lemma

Let $f_\varepsilon : S^2 \setminus \Delta \ni (x, y) \mapsto \frac{1}{|y - x + \varepsilon n(y)|^3} - \frac{1}{|y - x - \varepsilon n(y)|^3} dx$. *Then* $\exists \eta > 0, \exists M > 0,$ $\forall \alpha \in (-0.5, 3.5), \forall \varepsilon < \eta, \forall (x, y), |\varepsilon^\alpha f_\varepsilon(x, y)| \leq M \frac{1}{|x - y|^{5/2 - \alpha}}.$

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Optimization

We introduce the following costs:

- χ_B to ensure that we produce the magnetic field chosen :

$$\chi_B^2 = \int_P \langle B(x) \cdot n(x) \rangle^2 dS(x)$$

- A penalization term on j

$$\chi_j^2 = \int_S |j|^2 dS$$

$$\chi_{\nabla j}^2 = \int_S (|\nabla j_x|^2 + |\nabla j_y|^2 + |\nabla j_z|^2) dS.$$

- A penalizing term on the Laplace forces, for example $L^p(S, \mathbb{R}^3)$

$$\chi_F^2 = |L(j)|_{L^p} = \left(\int_S |L(j)|_2^p \right)^{1/p} dS$$

Thus, we will minimize the new cost with relative weights $\lambda_1, \lambda_2, \gamma \geq 0$.

$$\chi^2 = \chi_B^2 + \lambda_1 \chi_j^2 + \lambda_2 \chi_{\nabla j}^2 + \gamma \chi_F^2$$

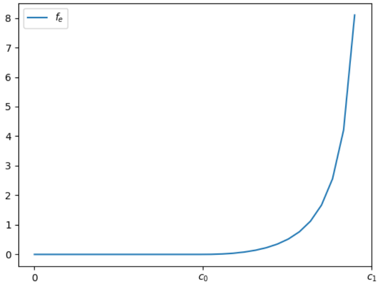
Lemma

Suppose $\lambda_1, \lambda_2, \gamma > 0$ *and* $p < \infty$ *then*

$$\inf_{j \in E} \chi_B^2 + \lambda_1 \chi_j^2 + \lambda_2 \chi_{\nabla j}^2 + \gamma |L(j)|_{L^p}$$

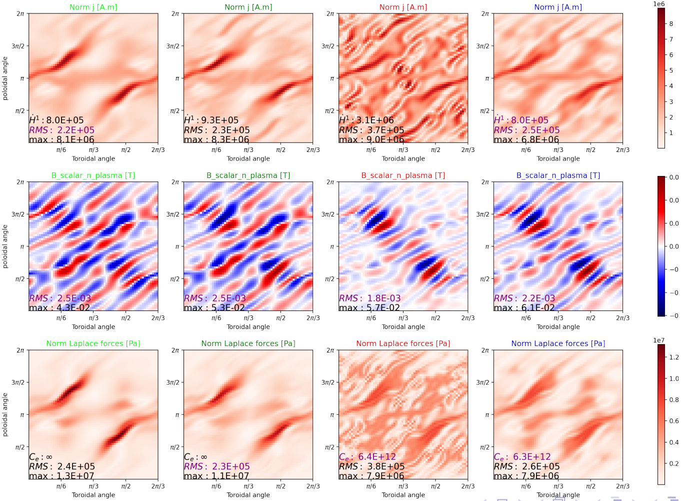
admit a minimizer.

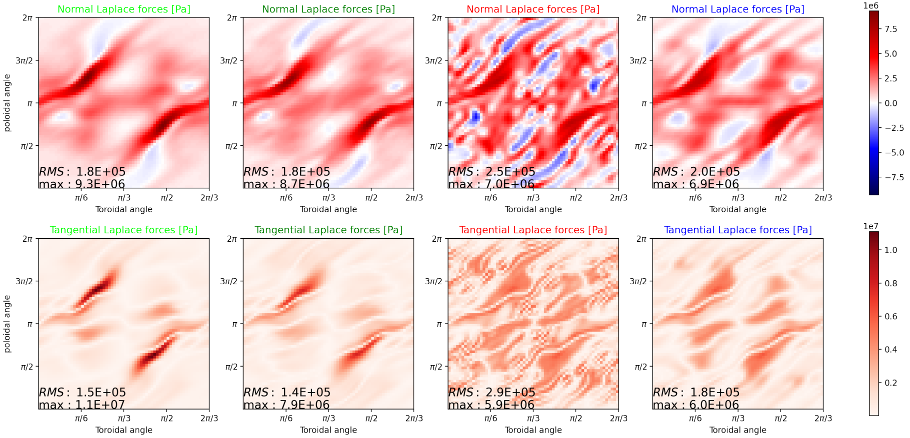
We also introduce a cost to penalize only high values of the forces:
$C_e = \int_S f_e(|L(j)|)$



Case	λ_1 $(T^2\ m^2/A^2)$	λ_2 $(T^2\ m^4/A^2)$	γ (T^2/Pa^2)	χ_F^2
1	$1.5 \cdot 10^{-16}$	0	0	0
2	0	0	10^{-17}	$\lvert L(j)\rvert^2_{L^2(S,\mathbb{R}^3)}$
3	0	0	10^{-16}	C_e
4	10^{-19}	10^{-19}	10^{-16}	C_e

(22)




Normal Laplace forces [Pa]
Normal Laplace forces [Pa]
Normal Laplace forces [Pa]
Normal Laplace forces [Pa]
poloidal angle
2π
3π/2
π
π/2
π/6
π/3
π/2
2π/3
Toroidal angle
RMS : 1.8E+05
max : 9.3E+06
RMS : 1.8E+05
max : 8.7E+06
RMS : 2.5E+05
max : 7.0E+06
RMS : 2.0E+05
max : 6.9E+06
1e6
7.5
5.0
2.5
0.0
−2.5
−5.0
−7.5
Tangential Laplace forces [Pa]
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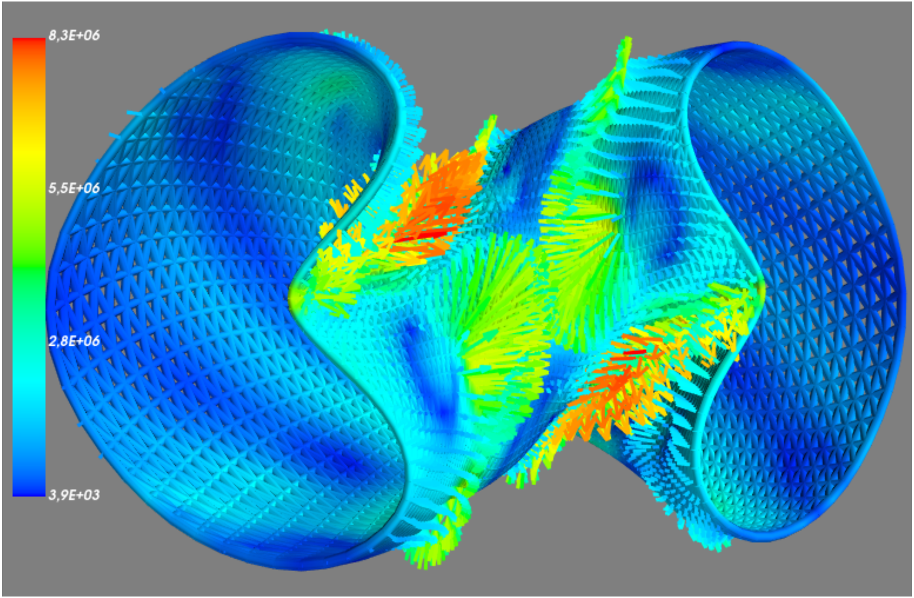
RMS : 1.5E+05
max : 1.1E+07
RMS : 1.4E+05
max : 7.9E+06
RMS : 2.9E+05
max : 5.9E+06
RMS : 1.8E+05
max : 6.0E+06
1e7
1.0
0.8
0.6
0.4
0.2


8,3E+06
5,5E+06
2,8E+06
3,9E+03

Prospects

- **Optimize** our implementation in order to provide a blackbox criteria which can easily be added to other optimization codes.
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Thank you for your attention !

Y. Han and M. Zhu.
Hardy-Littlewood-Sobolev inequalities on compact Riemannian manifolds and applications.
Journal of Differential Equations, 260(1):1–25, Jan. 2016.

M. Landreman.
An improved current potential method for fast computation of stellarator coil shapes.
Nuclear Fusion, 57(4):046003, Apr. 2017.
arXiv: 1609.04378.

P. Merkel.
Solution of stellarator boundary value problems with external currents.
Nuclear Fusion, 27(5):867–871, May 1987.
Publisher: IOP Publishing.