# Optimal Shape of Stellarators for Magnetic Confinement Fusion.

In collaboration with Yannick Privat [1] and Mario Sigalotti[2]

Rémi Robin,
Laboratoire Jacques-Louis Lions, Sorbonne Université, Paris, France

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[1]Université de Strasbourg
[2]Inria Paris

# Nuclear fusion confinement

- Goal : Confine a plasma of approx. 150 millions K for as long as possible with a density as high as possible in order to achieve fusion ignition.
- Solution : A plasma is made of ionized particules, thus interacts with a magnetic field.

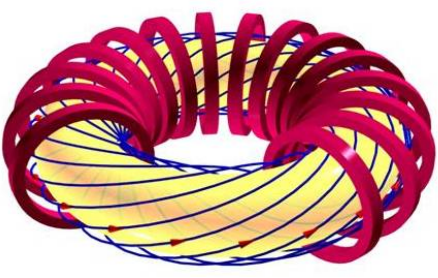

Figure: magnetic field lines inside a Tokamac, Inria team TONUS

# Stellarators

Stellarator approach : The magnetic confinement relies mainly on external coils.

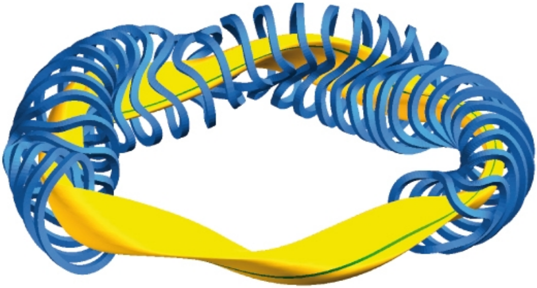

Figure: Wendelstein 7-X, Max-Planck Institut für Plasmaphysik

The plasma shape and the coils are obtained by several optimizations.

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2. We use a 'Coil winding surface' and find a current-sheet to generate the given $B_{target}$ [4].

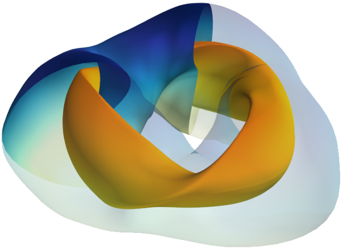

Figure: Coil winding surface and plasma surface of the NCSX Stellarator.

# Typical approach

1. Find a good magnetic field to ensure the plasma confinement.
2. We use a 'Coil winding surface' and find a current-sheet to generate the given $B_{target}$ [4].
3. (Approximate the current-sheet by several coils)

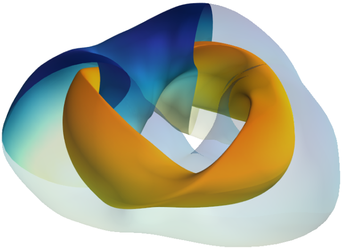

Figure: Coil winding surface and plasma surface of the NCSX Stellarator.

The magnetic field generated by the electric currents on the CWS (denoted S).

## Biot-Savart law in vacuo

$$\forall y \notin S, B(y) = \mathrm{BS}(j)(y) = \int_S j(x) \times \frac{y - x}{|y - x|^3} dS(x), \tag{1}$$

The figure of merit we use to ensure $B \approx B_{target}$ is

## plasma-shape objective

$$\chi_B^2(j) = \int_P |\mathrm{BS}(j)(y) - B_{target}(y)|^2 dy \tag{2}$$

## The goal

$$\inf_{\substack{j \in L^2(\mathfrak{X}(S)) \\ \operatorname{div} j = 0}} \chi_B^2(j) \tag{3}$$

## An inverse problem

$BS(\cdot)$ is continuous from $L^2(\mathfrak{X}(S)) \to C^k(\partial P, \mathbb{R}^3)$
$\implies j \mapsto BS(j)$ is compact (from $L^2(\mathfrak{X}(S)) \to L^2(P, \mathbb{R}^3)$).

- Use a finite dimensional subspace [4].
- Use a Tychonoff regularization [3].

$$\chi_j^2 = \int_S |j|^2 dS. \tag{4}$$

## Lemma

For any $\lambda > 0$, the problem

$$\inf_{\substack{j \in L^2(\mathfrak{X}(S)) \\ \operatorname{div} j = 0}} \chi_B^2 + \lambda \chi_j^2 \tag{P}$$

admits a unique minimizer

$$j_S = (\lambda \operatorname{Id} + \mathrm{BS}_S^\dagger \, \mathrm{BS}_S)^{-1} \, \mathrm{BS}_S^\dagger \, B_T. \tag{5}$$

# Table of Contents

We want to optimize on both the current sheet and the Coil Winding Surface.

## Admissible shapes

- Topology of a torus
- Regular enough
- Far enough to the plasma

## Shape optimization problem

$$\inf_{S \text{ admissible}} \left( \inf_{\substack{j \in L^2(\mathfrak{X}(S)) \\ \operatorname{div} j = 0}} \chi_B^2 + \lambda \chi_j^2 \right) \tag{SOP}$$

# Previous works

First approach by Paul, Abel, Landreman, Dorland [2019] [5]

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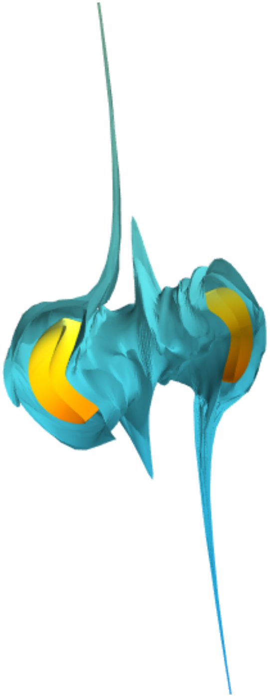
## Our contribution

- Existence of a minimizer of the shape optimisation problem,
- Computation of the shape gradient in the set of admissible shapes,
- Numerics based on our approach.

# Admissible shapes

Constraints on the set of admissible shapes $S \in \mathcal{O}_{\text{adm}}$:

1. S is a orientable surface homotopic to the usual torus.
2. $dist(S, P) \geq \delta$
3. $S$ is in included in a compact set
4. $\mathcal{H}^2(S) \leq A_M$



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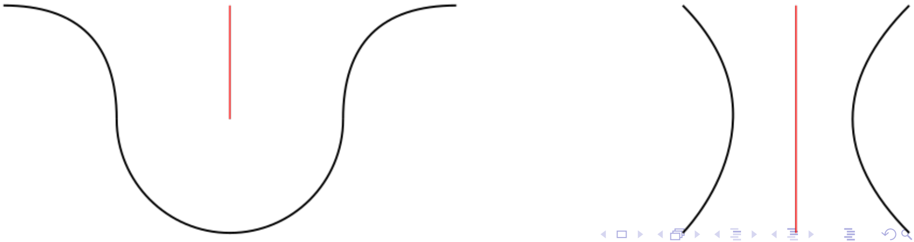
5. Lower bound on the reach of S

# Reach

[1][Delfour-Zolesio] and [2][1969 Federer]
$V \subset \mathbb{R}^n, \mathrm{Sk}(V)$ is the set of all points in $\mathbb{R}^n$ whose projection onto $V$ is not unique.

$$U_h(V) = \{x \mid d(x, V) < h\}$$

$$\mathrm{Reach}(V) = \sup\{h \mid U_h(V) \cap \mathrm{Sk}(V) = \emptyset\}$$



# Reach

**Theorem[2021, Privat, R. , Sigalotti]**

The shape optimisation problem

$$\inf_{S\in\mathcal{O}_{\text{adm}}} \inf_{\substack{j\in L^2(\mathfrak{X}(S))\\ \operatorname{div} j=0}} \chi_B^2 + \lambda\chi_j^2 \tag{6}$$

admits a minimizer.

Key ingredients of the proof :

- Compactness of $\mathcal{O}_{\text{adm}}$,
- Lower semicontinuity of the cost.
  - Transport $j$ while preserving tangent and divergence free,
  - Use a volumic approximation.

# Compactness

$$d_V(x) = \inf_{y \in V} |x - y|, \qquad b_V(x) = d_V(x) - d_{\mathbb{R}^3 \setminus V}(x);$$

Let $r$ be in $(0, r_{\min})$ and denote by $(S_n)_{n\in\mathbb{N}} = (\partial V_n)_{n\in\mathbb{N}}$ a sequence in $\mathcal{O}_{\text{adm}}$. Then, there exists $S_\infty = \partial V_\infty \in \mathcal{O}_{\text{adm}}$ such that, up to a subsequence,

- $b_{V_\infty}$ is in $\mathscr{C}^{1,1}(\overline{U_r(S_\infty)})$ and $(b_{V_n})_{n\in\mathbb{N}}$ converges to $b_{V_\infty}$ in $\mathscr{C}^1(\overline{U_r(S_\infty)})$;
- $(b_{V_n})_{n\in\mathbb{N}}$ converges to $b_{V_\infty}$ in $\mathscr{C}(\overline{D})$;
- $(d_{S_n})_{n\in\mathbb{N}}$ converges to $d_{S_\infty}$ in $\mathscr{C}(\overline{D})$;
- $(\mathscr{H}^2(S_n))_{n\in\mathbb{N}}$ converges to $\mathscr{H}^2(S_\infty)$.

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- $$\frac{\partial \tilde{C}(S, j_S)}{\partial S} = \frac{\partial \tilde{C}}{\partial S}(S, j_S) + \frac{\partial \tilde{C}}{\partial j} \frac{\partial j_S}{\partial S}(S, j_S).$$

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- The differential of $\varphi^\varepsilon = \mathrm{Id} + \varepsilon\theta$ provides a diffeomorphism from $\mathfrak{X}(S)$ to $\mathfrak{X}(S^\varepsilon)$.
- Nevertheless the range of $\mathscr{F}_S^0$ by $\varphi^\varepsilon$ does not coincide with $\mathscr{F}_{S^\varepsilon}^0$.

$$\Phi^\varepsilon : \mathscr{F}_S \longrightarrow \mathscr{F}_{S^\varepsilon}$$
$$X \longmapsto \frac{1}{[J(\mu_S, \mu_S^\varepsilon)\varphi^\varepsilon] \circ \varphi^{-\varepsilon}}(\mathrm{Id} + \varepsilon D\theta) X \circ \varphi^{-\varepsilon}$$

# Shape gradient

$$
\begin{aligned}
Z_P(k) &= \int_P K(\cdot, y) \times k(y)\, d\mu_P(y) \\
\widehat{Z}_P(k,j)(x) &= \int_P D_x \left( \frac{x-y}{|x-y|^3} \right)^T (k(y) \times j(x))\, d\mu_P(y), \qquad \forall x \in S.
\end{aligned}
$$

For every $\theta \in W^{2,\infty}(\mathbb{R}^3, \mathbb{R}^3)$ one has

$$
\langle dC(S), \theta \rangle = \int_S \theta \cdot (X_1 - \mathrm{div}_S (X_2)_{i:})\, d\mu_S \tag{7}
$$

where

$$
X_1 = -2\widehat{Z}_P(\mathrm{BS}_S\, j_S - B_T, j_S) \tag{8}
$$

$$
X_2 = -2Z_P(\mathrm{BS}_S\, j_S - B_T) j_S^T + 2\lambda j_S j_S^T - \lambda |j_S|^2 (I_3 - \nu \nu^T), \tag{9}
$$

where for $i \in \{1, 2, 3\}$, $(X_2)_{i:}$ denotes the $i$-th line of $X_2$ seen as a column vector, and $\nu$ denotes the outward normal vector on $S = \partial V$.

Numerical results for $\lambda = 2.5e^{-16}$

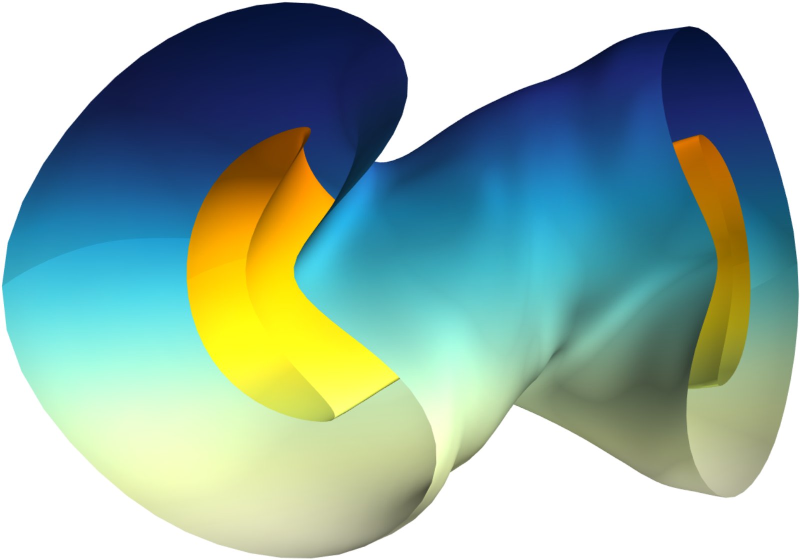
## Costs

| *Name* | $\chi_B^2$ | $\|B_{err}\|_\infty$ | $\chi_j^2$ | $\|j\|_\infty$ | *EMcost* |
|---|---|---|---|---|---|
| *ref* | $4.80 \cdot 10^{-3}$ | $5.15 \cdot 10^{-2}$ | $1.43 \cdot 10^{14}$ | $7.42 \cdot 10^{6}$ | $4.06 \cdot 10^{-2}$ |
| *DPC* | $1.23 \cdot 10^{-3}$ | $3.20 \cdot 10^{-2}$ | $9.48 \cdot 10^{13}$ | $5.99 \cdot 10^{6}$ | $2.49 \cdot 10^{-2}$ |

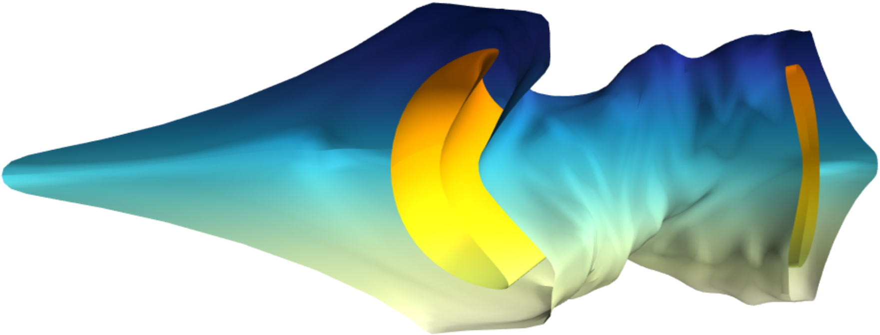
## Geometry

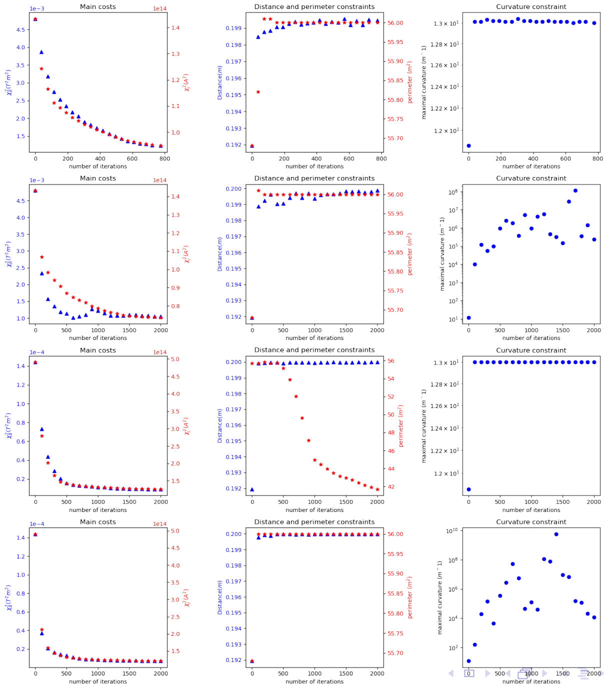
| *Name* | Distance ($m$) | Perimeter ($m^2$) | Maximal curvature ($m^{-1}$) |
|---|---|---|---|
| *Ref* | $1.92 \cdot 10^{-1}$ | $5.57 \cdot 10^{1}$ | $1.19 \cdot 10^{1}$ |
| *DPC* | $1.99 \cdot 10^{-1}$ | $5.60 \cdot 10^{1}$ | $1.30 \cdot 10^{1}$ |

# Reference



# After optimization




Main costs
Distance and perimeter constraints
Curvature constraint
number of iterations
Distance(m)
perimeter ($m^2$)
maximal curvature ($m^{-1}$)
$\chi^2_B (T^2 m^2)$
$\chi^2_j (A^2)$

# Prospects

- Collaboration with Renaissance fusion for industrial applications,
- Use more complex costs in the shape optimisation.

Thank you for your attention !

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# Cohomology and divergence free vector fields on the torus

## Hodge decomposition

On a closed Riemannian manifold $M$

$$L^2_p(M) = B_p \oplus B_p^* \oplus \mathscr{H}_p,$$

where

- $B_p$ is the $L^2$-closure of $\{d\alpha \mid \alpha \in \Omega^{p-1}(M)\}$,
- $B_p^*$ is the $L^2$-closure of $\{d^*\beta \mid \beta \in \Omega^{p+1}(M)\}$ ($d^*$ is the coderivative),
- $\mathscr{H}_p$ is the set $\{\omega \in \Omega^p(M) \mid \Delta_H \omega = 0\}$ of harmonic $p$-forms with $\Delta_H$ the Hodge Laplacian.

Thus for a flat Torus $T$, we only need to characterizes $B_1^*(T)$ and $\mathscr{H}_1(T)$.

- $B_1^*(T)$ is the $L^2$-closure of the 1-forms $\frac{\partial \Phi}{\partial u} dv - \frac{\partial \Phi}{\partial v} du$ for $\Phi \in \mathscr{C}^\infty(T)$.
- $\mathscr{H}_1(T)$ is a two-dimensional vector space as $b_1 = 2$.
  $\mathscr{H}_1(T) = \{\lambda_1 du + \lambda_2 dv \mid (\lambda_1, \lambda_2) \in \mathbb{R}^2\}$.

# In vacuo Maxwell equations on a toroidal 3D domain

Let $P$ a be toroidal domain. Let $\Gamma$ be a toroidal loop inside $P$ and denote by $I_p$ the electric current-flux across any surface enclosed by $\Gamma$ (also equal to the circulation of $B$ along $\Gamma$).

## Lemma

*Let $B \in \mathcal{C}^\infty(P, \mathbb{R}^3)$ such that $\operatorname{div} B = 0$ and $\operatorname{curl} B = 0$ in $P$. Let $g$ be the normal magnetic field on $\partial P$. Then $g$ and $I_p$ determine completely the magnetic field $B$ in $P$. Besides, there exists a constant $C > 0$ such that for every other magnetic field $\tilde{B}$ with the same total poloidal currents, $|B - \tilde{B}|_{L^2(P,\mathbb{R}^3)} \leq C|g - \tilde{g}|_{L^2(\partial P)}$ where $\tilde{g}$ is the normal component of $\tilde{B}|_{\partial P}$.*

Idea: consider the cochain complex

$$\mathscr{C}^\infty(P) \xrightarrow{\text{grad}} \mathscr{C}^\infty(P, \mathbb{R}^3) \xrightarrow{\text{curl}} \mathscr{C}^\infty(P, \mathbb{R}^3) \xrightarrow{\text{div}} \mathscr{C}^\infty(P).$$