Existence of surfaces optimizing geometric and PDE shape functionals under reach constraint

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A shape functional

For Ω regular enough,

$$F(\Omega) = \int_{\partial\Omega} j(x, \nu_{\partial\Omega}(x), B_{\partial\Omega}(x))\, d\mu_{\partial\Omega}(x),$$

- $\nu_{\partial\Omega}$ is the normal outward vector,
- $B_{\partial\Omega}(x)$ is either a geometric quantity (mean curvature, Gauss curvature ...) or the solution of a PDE defined on Ω or $\partial\Omega$.

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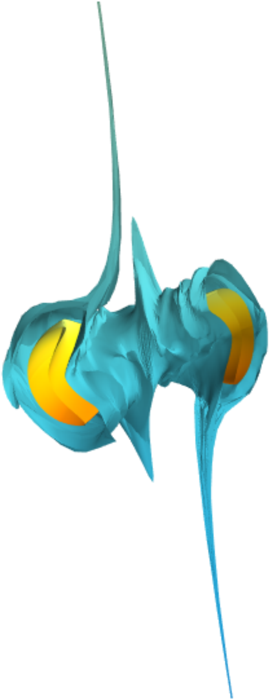
Existence of minimisers

Can we find $\Omega^* \in \mathcal{O}_{\text{adm}}$ such that

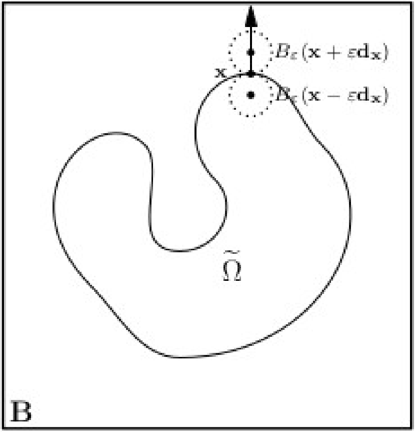
$$F(\Omega^*) = \inf_{\Omega \in \mathcal{O}_{\text{adm}}} F(\Omega)?$$



Uniform ball property

$\Omega \in \mathscr{O}_{r_0}$ if an only if $\Omega \subset D$ compact, $\forall x \in \partial\Omega$,

$$\exists d_x \in \mathbb{R}^n \mid \|d_x\|_{\mathbb{R}^d} = 1,\ B_{r_0}(x - r_0 d_x) \subset \Omega \text{ and } B_{r_0}(x + r_0 d_x) \subset \mathbb{R}^n \backslash \Omega.$$



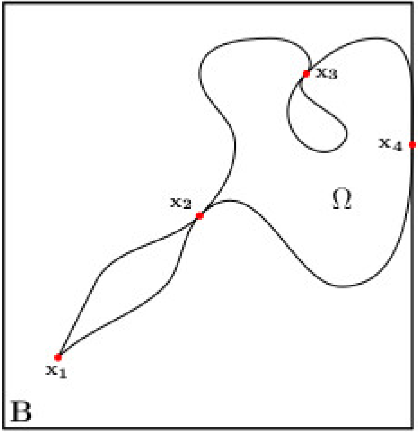


Figure taken from [Dal18].

Existing results

Theorem (Guo-Yang, 2013)

Let j be a continuous function from $\mathbb{R}^d \times \mathcal{S}^{d-1}$ to $\mathbb{R}$, then the following optimization problem

$$\inf_{\Omega \in \mathscr{O}_{r_0}} \int_{\partial\Omega} j(x, \nu(x)) d\mu_{\partial\Omega}(x)$$

admits a minimiser.

Theorem (Dalphin, 2018)

Let j be a continuous function from $\mathbb{R}^d \times \mathcal{S}^{d-1} \times \mathbb{R}$ and convex with respect to the last variable, then the following optimization problem

$$\inf_{\Omega \in \mathscr{O}_{r_0}} \int_{\partial\Omega} j(x, \nu(x), H_{\partial\Omega}(x)) d\mu_{\partial\Omega}(x)$$

admits a minimiser.

Let $h \in L^2(D)$, $g \in H^2(D)$, and define u_Ω as the solution of

$$\begin{cases} \Delta u_\Omega = h & \text{in } \Omega, \\ u_\Omega = g & \text{in } \partial\Omega. \end{cases}$$

Theorem (Dalphin, 2020)

Let j be a continuous function from $\mathbb{R}^d \times \mathcal{S}^{d-1} \times \mathbb{R} \times \mathbb{R}^d$, then the following optimization problem

$$\inf_{\Omega \in \mathscr{O}_{r_0}} \int_{\partial\Omega} j(x, \nu(x), u_\Omega(x), \nabla u_\Omega(x))\, d\mu_{\partial\Omega}(x)$$

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The direct method of calculus of variations

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2. Take a minimizing sequence and use a compactness result
3. Prove the lower-semicontinuity of the functional

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Characterization of a set

Distances functions

$$d_\Omega(x) = \inf_{y \in \Omega} \|x - y\|$$

$$b_\Omega(x) = d_\Omega(x) - d_{\mathbb{R}^d \setminus \Omega}(x)$$

Some properties

- For $x \in \partial\Omega$, $\nabla b_\Omega(x)$ is the unit outward normal vector,
- For $x \in \partial\Omega$, $\mathrm{Tr}(\nabla^2 b_\Omega(x))$ is the mean curvature,
- etc.

Uniform reach property

Definition

$$\text{Reach}(\Omega) = \sup\{h > 0 \mid d_\Omega \text{ is differentiable in } U_h(\Omega) \setminus \Omega\}.$$

Assume $\text{Reach}(\partial\Omega) = r_0 > 0$, we have

- if $\mathcal{H}^d(\partial\Omega) = 0$, then $\partial\Omega$ is a $\mathscr{C}^{1,1}$ hypersurface of $\mathbb{R}^d$ and satisfies the uniform ball property.
- For $h < r_0$, ∇b_Ω is $\frac{2}{r_0 - h}$-Lipschitz continuous on the tubular neighborhood $U_h(\partial\Omega)$.
- The restriction of ∇b_Ω to $\partial\Omega$ is $\frac{1}{r_0}$-Lipschitz continuous.

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R-convergence in $\mathscr{O}_{r_0}$

Given $(\Omega_n)_{n\in\mathbb{N}} \in \mathscr{O}_{r_0}^{\mathbb{N}}$, we say that $(\Omega_n)_{n\in\mathbb{N}}$ R-converges to $\Omega_\infty \in \mathscr{O}_{r_0}$ and we write $\Omega_n \xrightarrow{R} \Omega_\infty$ if

$$b_{\Omega_n} \to b_{\Omega_\infty} \quad \begin{cases} \text{in } \mathscr{C}(\overline{D}), \\ \text{in } \mathscr{C}^{1,\alpha}(U_r(\partial\Omega_\infty)), \forall r < r_0, \forall \alpha \in [0,1), \\ \text{weakly-star in } W^{2,\infty}(U_r(\partial\Omega_\infty)), \forall r < r_0. \end{cases}$$

Theorem

$\mathscr{O}_{r_0}$ is sequentially compact for the R-convergence.

Tubular neighboorhood

For $0 < h < r_0$, consider

$$T_{\partial\Omega}: \begin{array}{rcl} (-h,h)\times\partial\Omega & \to & U_h(\partial\Omega) \\ (t,x) & \mapsto & x + t\nabla b_\Omega(x). \end{array}$$

Since $T_{\partial\Omega}$ is Lipschitz continuous, it is differentiable at almost every (t_0, x_0), with

$$d_{(t_0,x_0)}T_{\partial\Omega}(s,y) = y + s\nabla b_\Omega(x_0) + t_0 d_{x_0}\nabla b_\Omega(y), \qquad \forall (s,y)\in\mathbb{R}\times T_{x_0}\partial\Omega.$$

Lemma

For every $\varepsilon > 0$, there exists $h > 0$ such that for all $\Omega \in \mathscr{O}_{r_0}$,

$$1-\varepsilon \leq \det(d_{(t_0,x_0)}T_{\partial\Omega}) \leq 1+\varepsilon, \quad \textit{for a.e. } (t_0,x_0)\in(-h,h)\times\partial\Omega.$$

Lemma

If $\Omega_n \xrightarrow{R} \Omega_\infty$ *then*

1. $\mathcal{H}^{d-1}(\partial\Omega_n)$ *converges toward* $\mathcal{H}^{d-1}(\partial\Omega_\infty)$ *as* $n \to +\infty$.
2. $\mathcal{H}^{d}(\Omega_n)$ *converges toward* $\mathcal{H}^{d}(\Omega_\infty)$ *as* $n \to +\infty$.
3. *If all the* $\partial\Omega_n$ *belong to the same isotopic class, then* $\partial\Omega_\infty$ *also belongs such a class.*

Corollary

$$\{\Omega \in \mathscr{O}_{r_0} \mid a \leq \mathcal{H}^{d-1}(\partial\Omega) \leq b, \, \partial\Omega \text{ is isotopic to } \partial\Omega_0\}$$

is sequentially compact

$$\mathcal{H}^{d-1}(\partial\Omega_n) = \int_{\partial\Omega_n} d\mu_{\partial\Omega_n}(x) = \frac{1}{2h}\int_{U_h(\partial\Omega_n)} \det(dT_n)\,dy$$

$$= \frac{1}{2h}\int_{U_{h-t}(\partial\Omega_\infty)} \det(dT_n)\,dy + \frac{1}{2h}\int_{U_h(\partial\Omega_n)\setminus U_{h-t}(\partial\Omega_\infty)} \det(dT_n)\,dy$$

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For any h, t small enough, there exists N_0 such that for all $n \geq N_0$,

$$\mathcal{H}^{d-1}(\partial\Omega_n) = (\mathcal{H}^{d-1}(\partial\Omega_\infty) + O(h))\left(1 + \mathrm{O}\left(\frac{t}{h}\right)\right) + \left(\mathrm{o}_{h\to 0}(1) + \mathrm{O}\left(\frac{t}{h}\right)\right)$$

Let j be a continuous function from $\mathbb{R}^d \times \mathcal{S}^{d-1} \times \mathbb{R}$ and convex with respect to the last variable.

$$F(\Omega) = \int_{\partial\Omega} j(x, \nu(x), H_{\partial\Omega}(x)) d\mu_{\partial\Omega}(x)$$

Theorem

F is a lower-semicontinuous shape functional for the R-convergence, i.e., for every sequence $(\Omega_n)_{n\in\mathbb{N}} \in \mathscr{O}_{r_0}^{\mathbb{N}}$ that R-converges toward Ω_∞, one has

$$\liminf_{n\to+\infty} F(\Omega_n) \geq F(\Omega_\infty).$$

As a consequence, the shape optimization problem

$$\inf_{\Omega\in\mathscr{O}_{r_0}} F(\Omega)$$

has a solution.

$$
\begin{aligned}
F(\Omega_n) &= \int_{\partial\Omega_n} j(x, \nabla b_{\Omega_n}(x), H_{\partial\Omega_n}(p_n(y))) d\mu_{\partial\Omega_n}(x) \\
&= \frac{1}{2h} \int_{U_h(\partial\Omega_n)} j(p_n(y), \nabla b_{\Omega_n}(p_n(y)), H_{\partial\Omega_n}(p_n(y))) \det(d_{T_n^{-1}(y)} T_n)\, dy.
\end{aligned}
$$

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F(\Omega_n) =& \frac{1}{2h} \int_{U_{h-t}(\partial\Omega_\infty)} j(p_n(y), \nabla b_{\Omega_n}(p_n(y)), H_{\partial\Omega_n}(p_n(y))) \det(dT_n)\, dy \\
&+ \frac{1}{2h} \int_{U_h(\partial\Omega_n)\setminus U_{h-t}(\partial\Omega_\infty)} j(p_n(y), \nabla b_{\Omega_n}(p_n(y)), H_{\partial\Omega_n}(p_n(y))) \det(dT_n)\, dy.
\end{aligned}
$$

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Definition

Let $f \in \mathscr{C}^0(D)$. We consider $v_{\partial\Omega}$ the solution of the equation

$$\Delta_{\partial\Omega} v_{\partial\Omega}(x) = f(x) \quad \text{in } \partial\Omega,$$

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$$\Delta_{\partial\Omega} v_{\partial\Omega}(x) = f(x) \quad \text{in } \partial\Omega,$$

$v_{\partial\Omega}$ is the unique minimiser of

$$\mathscr{E}_{\partial\Omega} : H^1_*(\partial\Omega) \ni u \mapsto \frac{1}{2}\int_{\partial\Omega} |\nabla_{\partial\Omega} u(x)|^2 d\mu_{\partial\Omega} - \int_{\partial\Omega} f(x)u(x) d\mu_{\partial\Omega} \qquad (1)$$

Lemma

For any $\Omega \in \mathscr{O}_{r_0}$, Eq. (1) admits one and only one minimiser.

$$F(\Omega) = \int_{\partial\Omega} j(x, \nu(x), v_{\partial\Omega}(x), \nabla_{\partial\Omega} v_{\partial\Omega}(x))\, d\mu_{\partial\Omega}(x),$$

where $j : \mathbb{R}^d \times \mathcal{S}^{d-1} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is assumed to be continuous.

Theorem (Privat-R-Sigalotti, 2022)

The shape functional F is lower-semicontinuous for the R-convergence, i.e., for every sequence $(\Omega_n)_{n\in\mathbb{N}} \in \mathscr{O}_{r_0}^{\mathbb{N}}$ that R-converges toward Ω_∞, one has

$$\liminf_{n\to+\infty} F(\Omega_n) \geq F(\Omega_\infty). \tag{2}$$

As a consequence, the shape optimization problem

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1. Transport $v_{\partial\Omega_n}$ to $\partial\Omega_\infty$ thanks to the orthogonal projector on $\partial\Omega_n$

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3. Check that $v^* = v_{\partial\Omega_\infty}$.
4. Passing to the limit is similar to the previous case.

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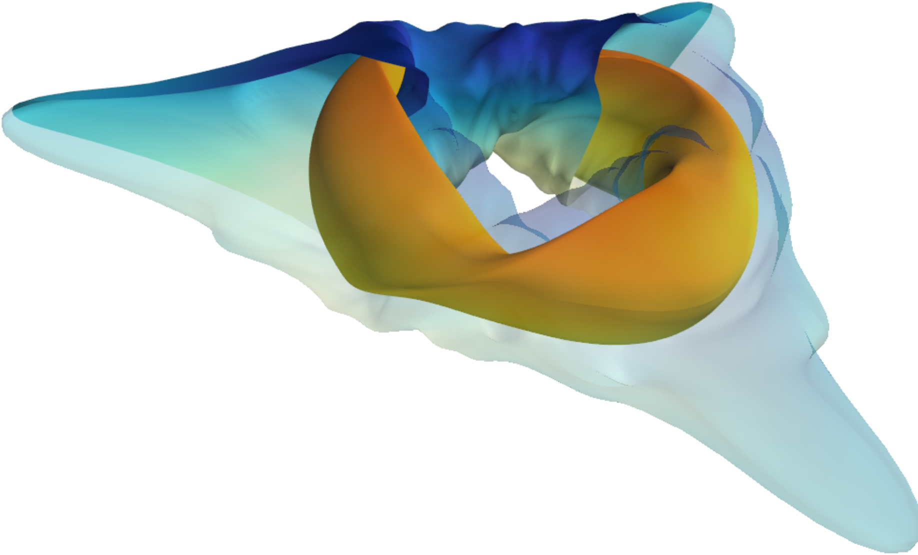

Conclusion

Hypersurfaces with a uniform Reach condition enjoy nice properties:

- Sequential compactness for the R-convergence.
- Many functionals involving geometric or PDE related cost are lower-semicontinuous for the R-convergence.
- Proofs are (relatively) straightforward.

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Thank you for your attention!

J. Dalphin. "Uniform ball property and existence of optimal shapes for a wide class of geometric functionals". In: *Interfaces Free Bound.* 20.2 (2018).

J. Dalphin. "Existence of optimal shapes under a uniform ball condition for geometric functionals involving boundary value problems". In: *ESAIM Control Optim. Calc. Var.* 26 (2020).

M. C. Delfour and J.-P. Zolésio. *Shapes and geometries*. Second. Vol. 22. Advances in Design and Control. Metrics, analysis, differential calculus, and optimization. 2011.

B.-Z. Guo and D.-H. Yang. "On convergence of boundary Hausdorff measure and application to a boundary shape optimization problem". In: *SIAM J. Control Optim.* 51.1 (2013).

Y. Privat, R. Robin, and M. Sigalotti. "Optimal Shape of Stellarators for Magnetic Confinement Fusion". In: *J. Math. Pures Appl. (9)* (2022).

Y. Privat, R. Robin, and M. Sigalotti. *Existence of surfaces optimizing geometric and PDE shape functionals under reach constraint*. soon.

[Dal20] [GY13] [DZ11] [PRSon] [PRS22]