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# About the minimization of magnetic forces on Stellarator coils

## Background and motivations

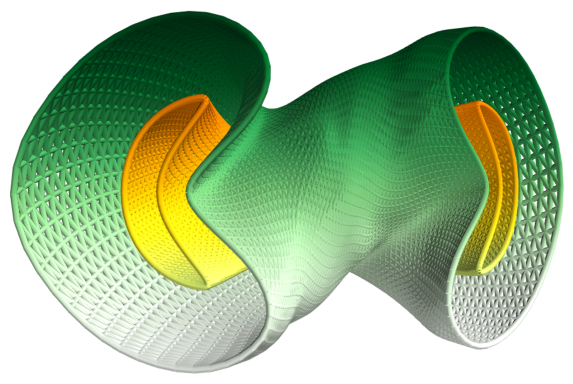

Fig 0 - Plasma surface (Yellow) and Coil Winding Surface (green) in NCSX design.

- Magnetic confinement devices for **nuclear fusion** can be large and expensive. Compact stellarators are promising candidates for cost-reduction, but introduce new difficulties: confinement in smaller volumes requires higher magnetic field, which calls for higher coil-currents and ultimately causes **higher Laplace forces** on the coils.
- In the present poster we consider a **coil winding surface**, we prove that there is a natural and rigorous way to define the **Laplace forces** on the current-sheet, and we provide numerical examples for the **NCSX** (now QUASAR) design.

## 1. The usual optimization problem for stellarator coils

Stellarator optimization codes typically involve a **2 steps** optimization :

1. Optimize the external magnetic field to improve the plasma confinement properties. The magnetic field is (nearly) entirely characterize by the **plasma surface** (a close toroidal surface such that the magnetic field is tangent).
2. Optimize the coils to :
   - Generate the magnetic field found in the first step.
   - Optimize the cost and feasibility of the coils.

We focus only on the second step, and suppose the plasma surface given. We also fix a **coil winding surface (CWS)** and we want to optimize among **tangent diverge-free vector fields** on the CWS.

### Magnetic field expression

Inside a stellarator, B is (in good approximation) only generated by currents on the CWS $S$. The field is given by the Biot-Savart law in vacuo:

$$\forall \mathbf{y} \notin S, \mathbf{B}(\mathbf{y}) = \mathbf{BS}(\mathbf{j})(\mathbf{y}) = \int_S \mathbf{j}(\mathbf{x}) \times \frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|^3} dS(\mathbf{x}), \quad (1)$$

### Inverse problem

To ensure that a current field $\mathbf{j}$ generate the objective magnetic field, we minimize the **plasma-shape objective** on the plasma surface $S_P$.

$$\chi_B^2 = \int_{S_P} \langle \mathbf{B}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) \rangle^2 dS(\mathbf{x}). \quad (2)$$

Mathematically, the operator $\mathbf{j} \mapsto \langle \mathbf{BS}(\mathbf{j}) \cdot \mathbf{n} \rangle$ is compact (from $L^2(\mathfrak{X}(S)) \to L^2(S_P)$). Finding the current-sheet $\mathbf{j}$ knowing the target magnetic field is thus an inverse problem. Several solutions have been proposed:

- Use a finite dimensional subspace for the space of vector field to solve the problem and use the dimension as a regularization parameter. See NESCOIL [1].
- Use a Tychonoff regularization $\lambda \chi_{\mathbf{j}}^2$ to ensure existence of the minimizer. This is done in REGCOIL code [2].

$$\chi_{\mathbf{j}}^2 = \int_S |\mathbf{j}|^2 dS. \quad (3)$$

## 2. Laplace forces on the CWS : definition and motivations

### Definition

The usual Laplace forces (exert by 3D current density) j is

$$\mathbf{F} = \mathbf{j} \times \mathbf{B}. \quad (4)$$

⚠**B is not defined on** $S$ :

A surface-current causes a **discontinuity** in the tangential component of the magnetic field, given by the interface condition $\mathbf{n}_{12} \times (\mathbf{B_2} - \mathbf{B_1}) = \mathbf{j}$. From a mathematical point of view, it is a consequence of the fact:

$$\forall \mathbf{y} \in S, \int_S \frac{1}{|\mathbf{x}-\mathbf{y}|^2} dS(\mathbf{x}) = +\infty. \quad (5)$$

- The resulting jump in the tangential component of B results in a normal force wherever $\mathbf{j} \neq 0$. That force, proportional to $|\mathbf{j}|^2$, tries to increase the thickness of the CWS.
- We focus on the other contributions to the Laplace force. We define them in a location $\mathbf{y} \in S$ as follows:

$$\mathbf{L}(\mathbf{j})(\mathbf{y}) = \lim_{\varepsilon \to 0} \frac{1}{2} \{\mathbf{j}(\mathbf{y}) \times [\mathbf{B}(\mathbf{y} + \varepsilon \mathbf{n}(\mathbf{y})) + \mathbf{B}(\mathbf{y} - \varepsilon \mathbf{n}(\mathbf{y}))]\}. \quad (6)$$

We will denote $\mathbf{L}_\varepsilon(\mathbf{j})(\mathbf{y})$ the expression inside the limit.

### Questions raised:

- Under which assumptions on j can we ensure that the **limit is well defined**?
- Can we find an **explicit expression** of L(j) (i.e., without a limit on $\varepsilon$)?
- Which functional space does L(j) belong to (for j in a given functional space)?

## 3. Theorem : existence and expression of the Laplace forces exerted on a current-sheet by itself

**Notation** Let $S$ be a smooth surface of $\mathbb{R}^3$, diffeomorphic to the 2-torus. We will denote $H^1(\mathfrak{X}(S))$ the completion of the smooth vector fields on $S$ for the $H^1(S, \mathbb{R}^3)$ norm. Let $\pi$ be the projector on the tangent bundle.

**Theorem** Let $\mathbf{j} \in H^1(\mathfrak{X}(S))$. Then $\mathbf{L}_\varepsilon(\mathbf{j})$ has an $\varepsilon \to 0$ limit in $L^p(S, \mathbb{R}^3)$, for any $1 \leq p < \infty$. Furthermore, $\mathbf{L}$ is a continuous (bilinear) map $H^1(\mathfrak{X}(S)) \to L^p(S, \mathbb{R}^3)$ given by

$$\mathbf{L}(\mathbf{j})(\mathbf{y}) = -\int_S \frac{1}{|\mathbf{y}-\mathbf{x}|} \left[ \mathrm{div}_\mathbf{x}(\pi_\mathbf{x} \mathbf{j}(\mathbf{y})) + \pi_\mathbf{x} \mathbf{j}(\mathbf{y}) \cdot \nabla_\mathbf{x} \right] \mathbf{j}(\mathbf{x}) d\mathbf{x} \quad (7)$$

$$+ \int_S \langle \mathbf{j}(\mathbf{y}) \cdot \mathbf{n}(\mathbf{x}) \rangle \frac{\langle \mathbf{y} - \mathbf{x} \cdot \mathbf{n}(\mathbf{x}) \rangle}{|\mathbf{y}-\mathbf{x}|^3} \mathbf{j}(\mathbf{x}) d\mathbf{x} \quad (8)$$

$$+ \int_S \frac{1}{|\mathbf{y}-\mathbf{x}|} \left[ \langle \mathbf{j}(\mathbf{y}) \cdot \mathbf{j}(\mathbf{x}) \rangle \, \mathrm{div}_\mathbf{x}(\pi_\mathbf{x}) + \nabla_\mathbf{x} \langle \mathbf{j}(\mathbf{y}) \cdot \mathbf{j}(\mathbf{x}) \rangle \right] d\mathbf{x} \quad (9)$$

$$- \int_S \langle \mathbf{j}(\mathbf{y}) \cdot \mathbf{j}_2(\mathbf{x}) \rangle \frac{\langle \mathbf{y} - \mathbf{x} \cdot \mathbf{n}(\mathbf{x}) \rangle}{|\mathbf{y}-\mathbf{x}|^3} \mathbf{n}(\mathbf{x}) d\mathbf{x} \quad (10)$$

### Proof ideas

- Note that $\frac{y-x}{|y-x|^3} = -\nabla_x \frac{1}{|y-x|}$.
- Do an integration by part on the tangential component of the gradient.
- Use some estimates when $\varepsilon$ is small to eliminate the part responsible for the magnetic discontinuity.
- Tools : Hardy-Littlewood-Sobolev inequality and Sobolev embeding on compact manifold [3].

## 4. Numerical results

We minimize the following figure of merits.

$$\chi^2 = \chi_B^2 + \lambda_1 \chi_\mathbf{j}^2 + +\lambda_2 \chi_{\nabla \mathbf{j}}^2 + \gamma \chi_F^2, \quad (11)$$

where $\chi_F^2$ is a "force objective" that penalizes strong forces on the current-sheet, i.e. among the coils. Possible definitions include:

$$\chi_F^2 = |\mathbf{L}(\mathbf{j})|^2_{L^2(S,\mathbb{R}^3)} = \int_S |\mathbf{L}(\mathbf{j})|^2 dS \quad (12)$$

$$\chi_F^2 = C_e = \int_S f_e(|\mathbf{L}(\mathbf{j})|) dS \quad (13)$$

with $f_e$ a non linear function penalizing high values.

We propose 4 simulations with different penalizations. The first one does not involve force minimization, the second takes into account the $L^2$ norm of the forces, the last two minimize peaks forces. Finally the last one has a $H^1$ regularization term. The weights chosen can be found in (14).

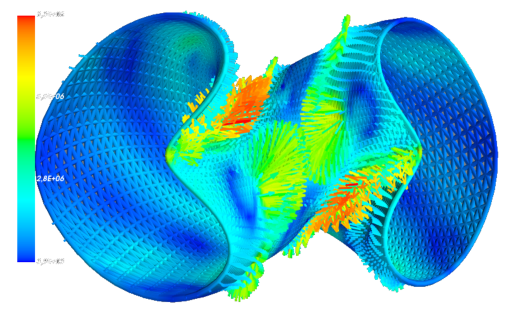

Fig 1 - The Laplace forces from the fourth simulation

| Case | $\lambda_1$ (T² m²/A²) | $\lambda_2$ (T² m⁴/A²) | $\gamma$ (T²/Pa²) | $\chi^2_F$ |
|---|---|---|---|---|
| 1 | $1.5 \cdot 10^{-16}$ | 0 | 0 | 0 |
| 2 | 0 | 0 | $10^{-17}$ | $\lvert\mathbf{L}(\mathbf{j})\rvert^2_{L^2(S,\mathbb{R}^3)}$ |
| 3 | 0 | 0 | $10^{-16}$ | $C_e$ |
| 4 | $10^{-19}$ | $10^{-19}$ | $10^{-16}$ | $C_e$ |

(14)

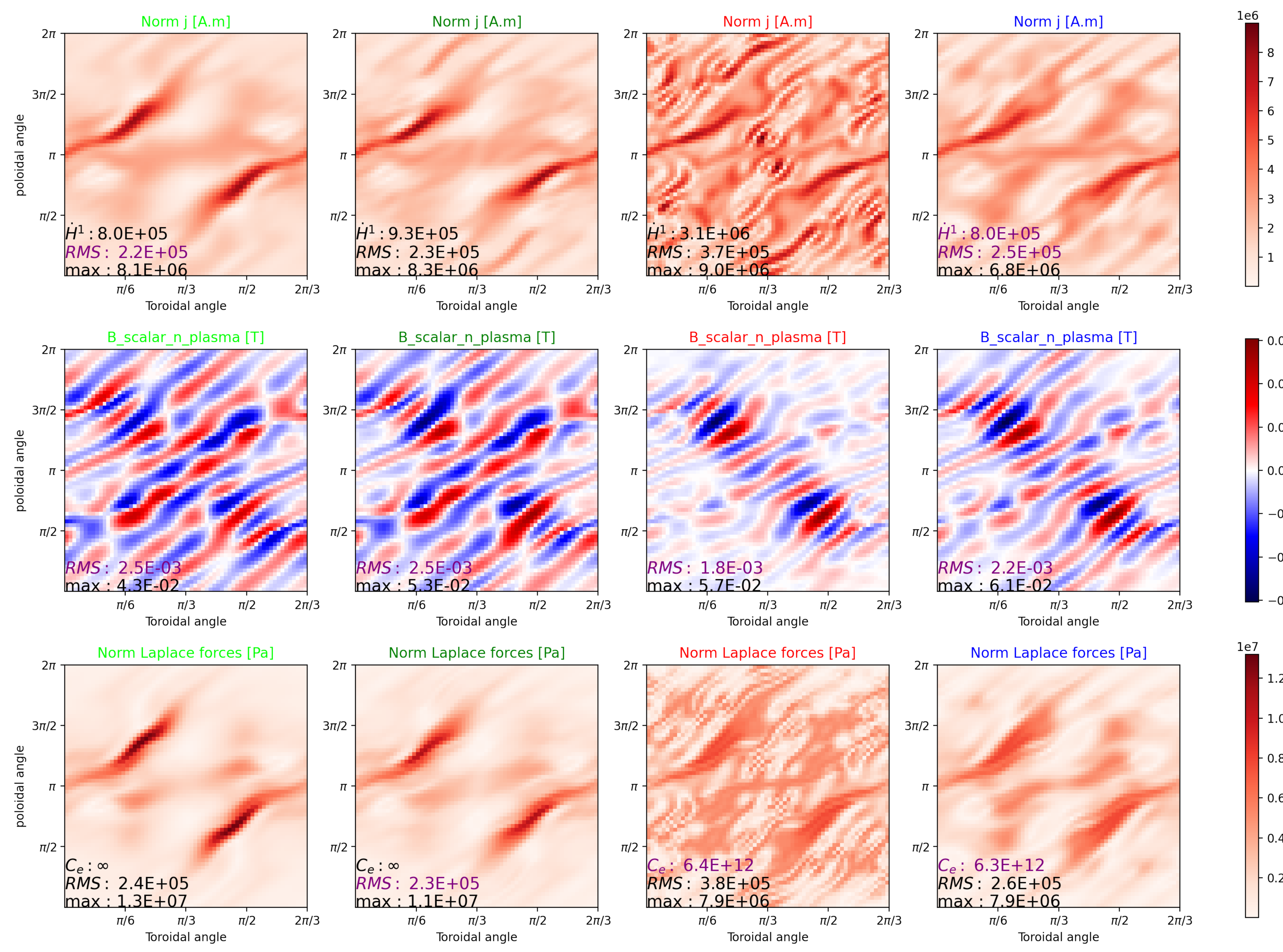


Fig 2 - Results for different penalizations on NCSX. 624 DOF are used for j on every simulation. RMS stands for the Root Mean Square surface average. We wrote in purple the quantities which are taken into account in the minimization on every column

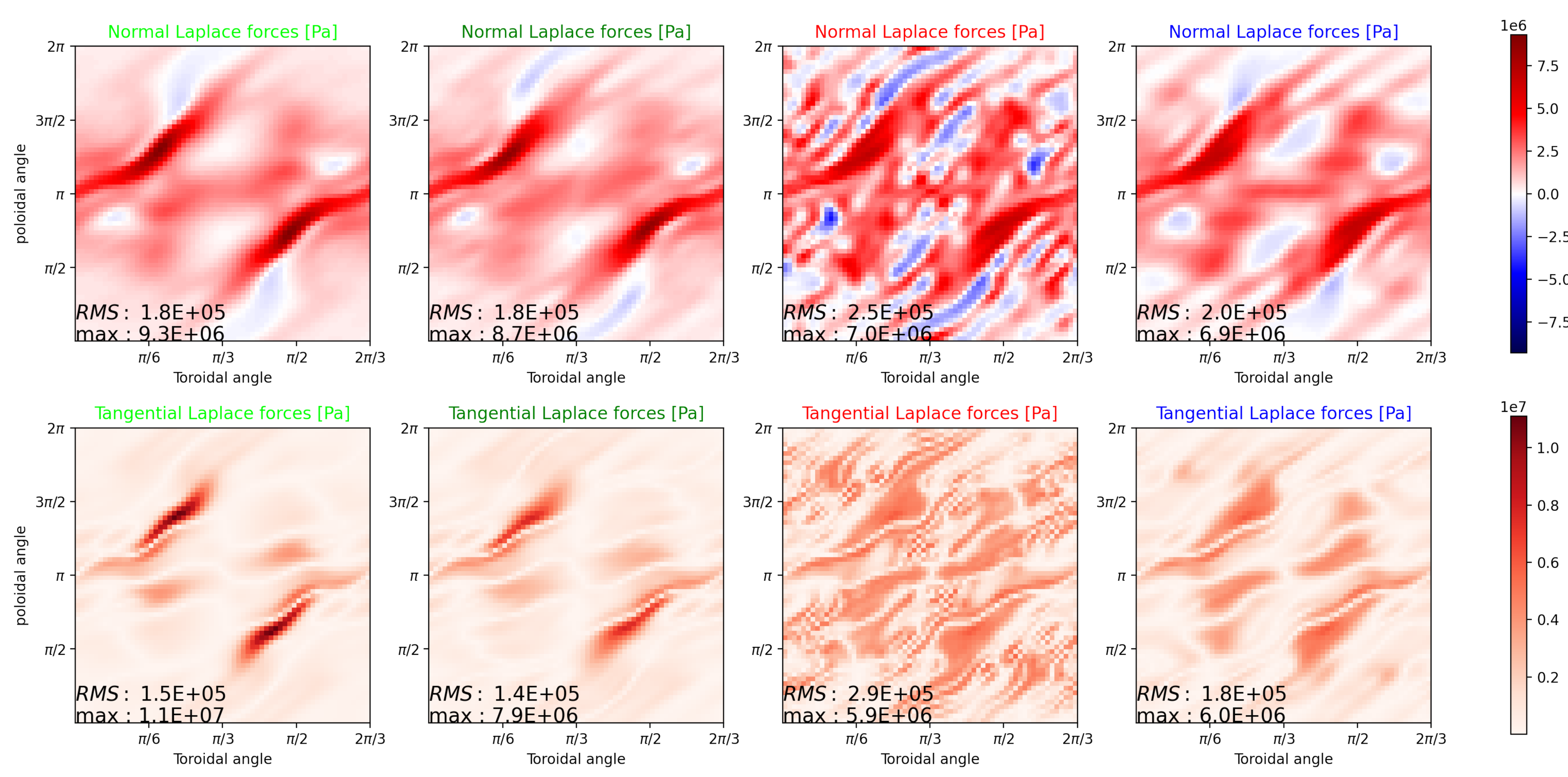


Fig 3 - Tangential and normal components of the Laplace forces of the simulations of Fig. 2

In conclusion it was possible to simultaneously optimize the NCSX coils for high magnetic fidelity, high regularity and low forces, e.g. 40% lower peak forces compare to REGCOIL for similar plasma shape accuracy, slightly higher average current and lower current peak.

## 5. Prospects and Acknowledgements

- **Optimize** our implementation in order to provide a blackbox criteria which can easily be added to other optimization codes.
- Use **shape optimization** on the CWS to improve the pareto optima.

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